CALIFORNIA INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

ME 96
Cantilever Beam Experiment

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1 Overview of experiment

The experiment involves the bending and vibration of an aluminum beam. Measurements are made of the deflection, strain rates, fundamental frequency, and damping constant. The student is exposed to measurement techniques, data acquisition, and analysis. The experimental results are also compared with theory.

2 Analysis of beam strain and deflection

This section briefly reviews the material necessary to make calculations of beam deflection and strain for an imposed load. The notation corresponds with the Figure (a).

Recall for a linearly elastic beam that the strain in the x–direction, ε_x, is linearly related to the imposed normal stress in the x–direction, σ_x, by the following reaction:

$$\varepsilon_x = \frac{\sigma_x}{E}, \tag{1}$$

where E is the modulus of elasticity.

In pure bending, the strain on a beam can be expressed in terms of the radius of curvature of the beam, R, and the distance from the neutral axis of the beam, y:

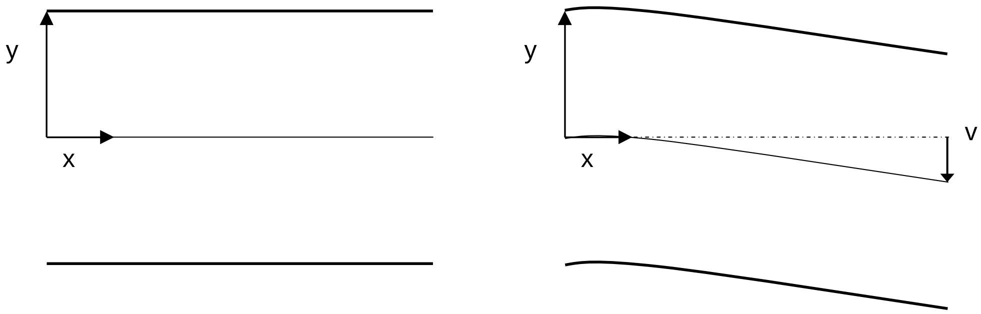
$$\varepsilon_x = -\frac{y}{R} \tag{2}$$

Hence, the strain is zero along the neutral axis and its magnitude increases with distance from the neutral axis. The strain is negative in the $+y$–direction, which corresponds to a negative stress or compression in the $+y$ part of the beam. Equations (1) and (2) can be rearranged to give the stress at some position within the beam,

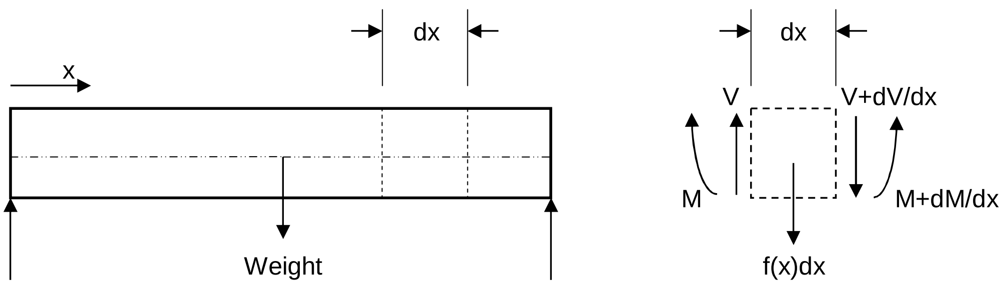
$$\sigma_x = -\frac{E_y}{R} \tag{3}$$

The moment, $M(x)$, acting on the beam can be calculated from the stress:

$$M(x) = -\int_A y\sigma_x dA \tag{4}$$



(a)



(b) Beam of uniform density

Force and moment balance within the beam

Hence, a positive moment produces compression in the $+y$ fibers of the beam. Recalling the definition for the moment of inertia, I:

$$I = \int_A y^2 dA, \tag{5}$$

the moment-curvature relation can be found for a homogeneous beam:

$$M = \frac{EI}{R}, \tag{6}$$

and the flexure formula follows as:

$$\sigma_x = -\frac{My}{I} \tag{7}$$

The radius of curvature of the beam can be related to the displacement, v, of the neutral axis of the beam due to bending. For small deflections of the beam compared to the length of the beam, the radius of curvature can be determined from the following equation:

$$\frac{1}{R} = \frac{\frac{d^2v}{dx^2}}{[1 + (\frac{d^2v}{dx^2})^2]^{3/2}} \tag{8}$$

This equation can be approximated as:

$$\frac{1}{R} = \frac{d^2 v}{dx^2} \tag{9}$$

3 Analysis of beam vibration

Consider a beam of uniform cross section, A, and density, r, as shown in the Figure (b). At some position x within the beam, a balance of forces and moments can be drawn using $f(x)$ as some distributed load on the beam.

The resulting equations for the balance of the force and moments are as follows (assuming that dx approaches zero):

$$-f(x) = \frac{dV}{dx} \tag{10}$$

and

$$V = \frac{dM}{dx} \tag{11}$$

These equations can be combined with the moment-deflection equation:

$$M = -EI\frac{\partial^2 v}{\partial x^2} \tag{12}$$

Note, here the deflection, $v(x)$, is defined in the opposite sense as used in equation (2). These equations can be combined to give:

$$EI\frac{\partial^4 v}{\partial x^4} = f(x) \tag{13}$$

The distributed load is the inertial load due to vibration and can be represented as the product of the mass per unit length and the cross sectional area in the direction opposite of the acceleration:

$$f(x) = -\rho A\frac{\partial^2 v}{\partial t^2} \tag{14}$$

Combining equations (4) and (5) results in the following fourth order partial differential equation:

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} = 0 \tag{15}$$

This differential equation can be solved subject to specified boundary and initial conditions.

The above analysis assumes that there is no damping of the vibrating beam, and hence the vibration of the beam would be of constant amplitude. However, in real systems damping is usually always present due to friction. In a vibrating beam the friction is internal to the medium, and is associated with the energy being dissipated randomly in the crystal lattice. The energy dissipation manifests itself as an internal heating of the beam. The frictional damping reduces the amplitude of vibration over time.

For a vibrating system with a single degree of freedom (harmonic oscillator), the motion of the system is governed by the following equation:

$$y = a_1 \cos pt + a_2 \sin pt, \tag{16}$$

where p is the frequency of vibration and is related to the period of vibration by $\tau = 2\pi/p$, and a_1 and a_2 are unknown constants. If the system is damped and the damping force is proportional to velocity (as is the case for internal friction), the motion of the damping system is governed by the following equation:

$$y = e^{-nt}(a_1 \sin qt + a_2 \cos qt), \tag{17}$$

where the n is the damping constant and vibratory motion now has the period:

$$\tau = \frac{2\pi}{q} = \frac{2\pi}{\sqrt{(p^2 - n^2)}} \tag{18}$$

Note that if n is small relative to p, the period of vibration is close to the value obtained without damping.

4 Analysis: calculation of fundamental frequency

Using the analysis above we want to determine the fundamental frequency of vibration for the beam used in the experiment (both axes). To find the value, it is useful to follow the following steps.

- Assume that separation of variables can be used to solve equation (15).
- Assume that the beam is undamped. Show that the solution for the vibration of the beam as a function of time takes the form given in equation (16).
- The solution of the differential equation for the x–variation is given by:

  $$b_1 \sin \lambda x + b_2 \cos \lambda x + b_3 \sinh \lambda x + b_4 \cosh \lambda x,$$

  where the b's are unknown constants. This solution can also be rewritten as:

  $$c_1(\cos \lambda x + \cosh \lambda x) + c_2(\cos \lambda x - \cosh \lambda x) \\ + c_3(\sin \lambda x + \sinh \lambda x) + c_4(\sin \lambda x - \sinh \lambda x),$$

  which is a useful representation for the present configuration.
- Using appropriate boundary conditions for the cantilever beam, find the fundamental frequency in terms of E, I, A, and ρ. You will need to know the zeros of the transcendental equation, $\cos \lambda L \cosh \lambda L = -1$, which are $\lambda L = (1.875, 4.694, 7.855, 10.996, ...)$. Note you do not need initial conditions to solve for the frequency.

5 Lab Report

You must address the following points in your report. However, do them in any sensible order- you don't need to stop in the middle of taking data to do a detailed comparison to theory.

1. Make a sketch in your notebook of the experimental setup, and give a short description of the major components. Include any relevant dimensions you will need for the analysis.

2. Apply different loads on the end of the beam and record the resulting strain and tip deflection. Do the measurements for both axes of the beam and for all five strain gauges. Plot your results.

3. Investigate the repeatability of these measurements. To do so, choose a load value and apply it multiple times to the beam, each time beginning from an unloaded state. Record the strain and deflection readings each time. Do at least six repetitions. Do this entire process for at least two loads, for example the largest load from step (1) and a moderate load. Repeat for the other beam axis.

4. Familiarize yourself with the oscilloscope and data acquisition system. Record the bandwidth, sampling rate, and resolution.

5. Using the oscilloscope and the output from any one strain gauge, determine the fundamental vibrational frequency of the beam on each axis. Make several measurements, so that you can estimate the repeatability. Note that the best accuracy is obtained by measuring times between zero-crossings, not between peaks or valleys.

6. Now measure the fundamental vibrational frequency for both axes using the data acquisition system and the FFT (Fast Fourier Transform) routine. Repeat several times.

7. Examine the effect of sampling rate and total number of samples on the FFT results. Include at least one measurement with a sampling rate less than twice the frequency of oscillation. Make a plot of the measured fundamental frequency vs. sampling rate.

8. To appreciate how much care must be taken with strain gauges (or for that matter with many types of delicate instruments), take one of the uninstrumented beams and apply strain gauges to it. Each student should apply at least one strain gauge. Repeat selected measurements of the static and dynamic response and compare the results. Do your results agree to within previously-determined uncertainties? If not, can you diagnose what went wrong with your strain gauge application? What evidence can you give that the discrepancy is due to one cause, and not some other?

9. Compare your measured strain vs. load data to theoretical predictions for a linearly elastic beam in pure bending. Show in a plot how the two compare, and include error bars on the measured values. Discuss any discrepancies found, including possible causes. (Were there unaccounted-for

systematic errors in the measurements? Does the theory make simplifying assumptions that aren't valid? How do you know? What additional measurements could you do to determine the problem?

10. Plot the tip deflection vs. load, and compare to the solution determined by integrating the differential equation. As in (1), discuss any discrepancies, their possible causes, and the steps that could be taken to resolve the discrepancies.

11. If the mass of the beam were taken into account, how would it affect the deflection vs. load data? (The density of aluminum is 2.77 g/cm^3.)

12. Time permitting, explore other effects of your own choosing. Some interesting questions: What is the effect of combined static and dynamic loading? Does it change the vibration frequency? What would the dynamic response look like if you were to hang a mass from a spring or a pendulum attached to the end? How would you analyze such a problem theoretically?

Prelab

- Derive equation (15) along the lines discussed, but including the effects of gravity. Assume the beam is horizontal.
- Solve this equation for the steady-state deflection due to gravity.
- Determine the fundamental frequency of vibration in terms of E, I, ρ, and A by solving (2-6) any way you like (with or without gravity). One approach is described in the text; numerical solution with MATLAB, Excel, or anything else is also fine.