BE/APh161: Physical Biology of the Cell
Homework 5
Due Date: Wednesday, Feb. 12, 2020

"One of the principal objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity." - Josiah Willard Gibbs

1. Elasticity, Hydrodynamics and Indicial Notation.

This problem aims to give you practice in thinking about indicial notation and gives you a chance to think further about the ideas concerning elasticity and hydrodynamics that we will spend several weeks on in class. A key notational convenience that will be afforded us is the use of the summation convention. The basic injunction is: *sum over all repeated indices*. So as to gain familiarity with this convention, work out the following examples.

(a) Write out $a_i b_i$ as a full sum and give its standard interpretation in vector analysis by writing it in vectorial form.

(b) The electric current density $\mathbf{j}$ is related to the applied electric field $\mathbf{E}$ through the relation $\mathbf{j} = \boldsymbol{\sigma}\mathbf{E}$. Write this relation in indicial notation and then do the sums. How many equations is this? Write them all out.

(c) The vector cross product can be written as $a_i = \epsilon_{ijk} b_j c_k$, where the Levi-Cevita symbol is defined as $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$ and 0 otherwise. Another way of stating this that the Levi-Cevita symbol is 1 for even permutations of ϵ_{123}, -1 for odd permutations and zero for all other cases. Using these conventions, show that the expression written in summation convention notation yields the correct components of the vector cross product.

(d) Write out $\partial(\rho v_i)/\partial x_i$ by following the edict of the summation convention.

(e) Rewrite ∇p in indicial notation. The gradient in pressure will be important to us when considering the Navier-Stokes equations.

(f) Write out $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ in indicial notation.

(g) When we balance forces in continuum mechanics, we will be interested in the divergence of the stress tensor. Write out $\partial \sigma_{ij} / \partial x_j$ using the summation convention. How many equations is this? Write them all out.

(h) Write out

$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}, \tag{1}$$

in indicial form and in vectorial form using ∇.

(i) Consider the matrix equation $\mathbf{a} = \mathbf{M}\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are column vectors with three components and $\mathbf{M}$ is a 3×3 matrix. Write out the rules for matrix multiplication for this problem in indicial notation.

(j) $\frac{\partial v_i}{\partial x_i}$ (write this in direct vectorial notation also).

(k) Given a matrix $\mathbf{M}$, what is M_{ii}? What is another way of writing this? Consider the matrices $\mathbf{A}$ and $\mathbf{B}$. Write the ij^{th} element of the matrix $\mathbf{AB}$ in terms of the matrix elements of $\mathbf{A}$ and $\mathbf{B}$ individually. Use indicial notation.

(l) In the Navier-Stokes equations one encounters terms like $\mathbf{v} \cdot \nabla \mathbf{v}$. Rewrite this in indicial notation, using the summation convention.

(m) In linear elasticity, the stress tensor is of the form $\sigma_{ij} = C_{ijkl} \epsilon_{kl}$. Write out the components of σ_{11} and σ_{12} of the stress tensor by exploiting the summation convention.

(n) The equilibrium equations for elasticity are written as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0. \tag{2}$$

b_i is the i^{th} component of the "body force" (e.g. gravity). This is three equations corresponding to $i = 1, 2, 3$. Write all three equations by using the summation convention.

(o) For the particular case of an isotropic, linear elastic solid, the elastic modulus tensor is of the form

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}). \tag{3}$$

In this case, find an expression for the stress $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ and the stored energy density of the solid, $W(\{\epsilon_{ij}\}) = \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl}$. Write your expression for the stress in both indicial and vector notation. Also, use this form for the elastic modulus tensor to obtain the equilibrium equations (the so-called Navier equations) by plugging your result for σ_{ij} into

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0. \tag{4}$$

Note that we are looking at the particular case in which the body force has been set to zero.

(p) The Navier-Stokes equations are of the form

$$\rho(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}) = \mu \frac{\partial^2 v_i}{\partial x_k \partial x_k} - \frac{\partial p}{\partial x_i}. \tag{5}$$

Write all three equations by exploiting what you know about the summation convention. Also, write these equations in direct (vectorial) form.

2. Equation of Motion for Mean Cytoskeletal Filament Length

In class we discussed the rate equation protocol shown in Figure 1. Our application of the protocol in class was to the problem of a constitutive promoter and provided a dynamical equation for the average number of mRNAs per cell as a function of time. In this problem, you are going to imitate that analysis, but this time thinking about the average length of a cytoskeletal filament as a function of time. Imagine a situation in which we have a closed box in which a single cytoskeletal filament has been nucleated (using a nucleating factor, for example) and which is bathed in a reservoir of monomers, with the initial number of monomers being given by N_{tot}. Our goal is to compute $L(t)$, where L is the length of the filament as a function of time. The rate at which monomers attach is $k_{on}n_{free}$, where n_{free} is the number of free monomers and the rate at which monomers detach from the tip of the growing filament is k_{off}. Write a dynamical growth equation for the dynamics of

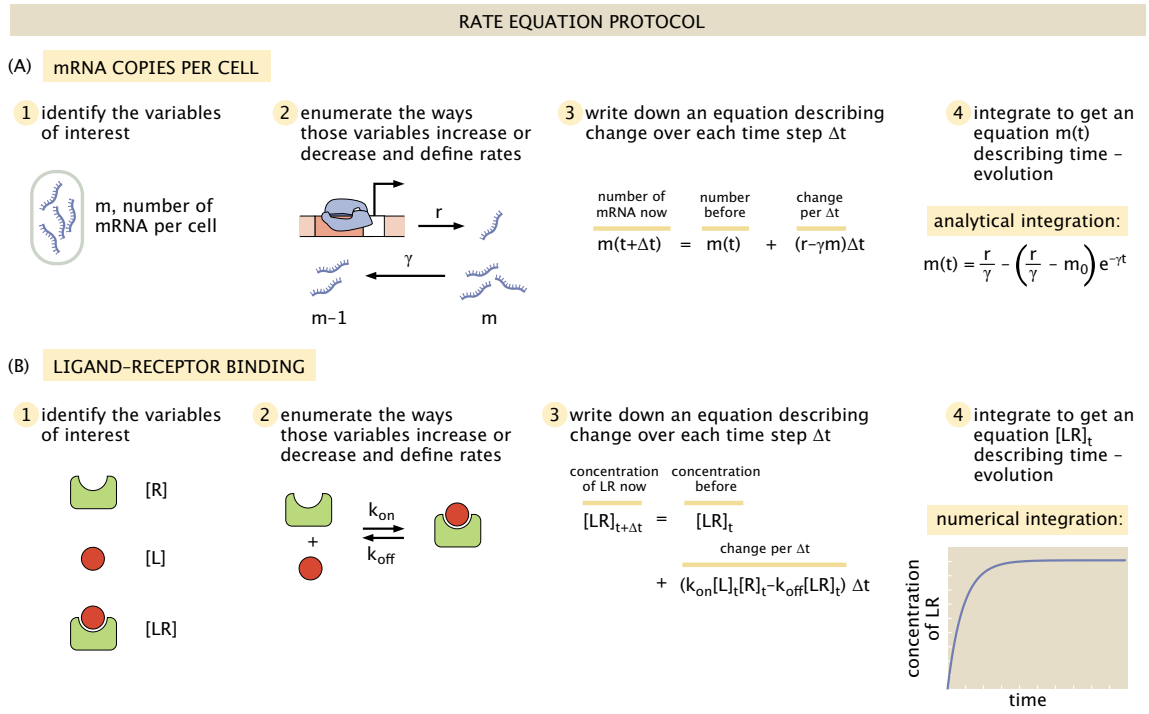

Figure 1: The rate equation protocol. To write dynamical equations for the time evolution of quantities of biological interest, there is a progression of steps.

$L(t)$ and find the solution. What is the steady-state length of the filament? Make a plot of the length as a function of time - you can attempt to figure out reasonable choices of the parameters by looking at book.bionumbers.org or by looking at PBoC, but give an explanation of your choices. Also, compare and contrast the analysis here with that done in class for the constitutive promoter.

3. Fluorescence Recovery After Photobleaching and Diffusion.

In class I introduced the experimental method known as FRAP (Fluorescence Recovery After Photobleaching). This technique is founded upon an annoying feature of fluorescent molecules, namely, that if you shine light on them for too long they stop giving off light. As often happens, people figured out how to turn this annoyance into something useful. In particular,

FRAP is often used to learn about the way that different parts of cells are in diffusive contact.

In this problem, I want you to carry out a full derivation of the concentration as a function of position and time after photobleaching a cell of radius 25 microns with a "hole" of radius 2 microns. Think of the cell as having a thickness of 1 micron. (Looking at the treatment of the one-dimensional version of this problem in chap. 13 of PBoC will be helpful. Also, this part of the problem is effectively problem 13.4 of PBoC2.) For simplicity, ignore the presence of a nucleus, think of the cell as a perfect circle and imagine the photobleached region as a circle at the center of the circular cell.

(a) Consider an initial concentration c_0 of the fluorescent molecule of interest which is uniformly distributed throughout the cell. How many molecules of the fluorescent molecule are there - write an equation that gives this number?

(b) Before doing any calculations, explain what the final concentration c_∞ will be after infinite time, when the system has returned to equilibrium. You may assume that once a molecule has been photobleached it is effectively dead and can be forgotten.

(c) Your goal now is to compute the recovery curve. What this means is that you need to work out how many fluorescent molecules are in the photobleached region as a function of time. Make graphs for the case where the photobleached region is centered about the origin. Make sure when you make your plots you use reasonable values for the diffusion constant - justify your choice.

(d) One of the uses of the FRAP technique is to determine the diffusion constant of various molecules within the cytoplasm of cells. Discuss how that might work on the basis of the derivation you have given here.

To do this problem you will need the table of zeros of the first derivative of $J_0(x)$ given in the file attached to the homework. Make sure you explain exactly what you are doing and what your results mean. Also, I want you to plot the results for the recovery curve for different number of terms kept in the Bessel series. Use just enough terms in the Bessel series such that your answer has 5% accuracy in the region of interest, namely, the FRAPed

region, and tell us how many terms you used.

Here is some stuff that will come in handy when thinking about this problem. To obtain the solution, exploit the method of separation of variables which posits a solution of the form $c(r,t) = \rho(r)T(t)$. The equation for ρ can be beaten into the form

$$\frac{d^2\rho}{dz^2} + \frac{1}{z}\frac{d\rho}{dz} + \rho = 0, \tag{6}$$

where z is an effective variable that arises in the separation of variables process. The only solution to this equation that does not diverge for $z \to 0$ is the zero order Bessel function $J_0(z)$. Next, the boundary conditions at the edge of the cell will lead to a condition of the form

$$J_0'(kR) = 0. \tag{7}$$

Interestingly, the roots of J_0' are just the roots of J_1 because of the identity $J_0'(z) = -J_1$. The full solution you are looking for will emerge as (make sure you demonstrate this clearly and convincingly)

$$c(r,t) = a_0 + \sum_{i=1}^{\infty} a_i e^{-Dk_i^2 t} J_0(K_i r). \tag{8}$$

We can determine the coefficients a_i using the initial condition $c(r,0)$. Another identity that will prove useful when doing the calculation of the coefficients is: $\int zJ_0(z)\,dz = zJ_1(z)$.